Inner and outer characterization of the projection of polynomial equations using symmetries, quotients and intervals

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Palaiseau, IMT

Motivation

Consider the set

$$\mathbb{X} = \{\mathbf{x} \in \mathbb{R}^n | \mathbf{f}(\mathbf{x}) = \mathbf{0}\}$$

where $\mathbf{f} : \mathbb{R}^n \mapsto \mathbb{R}^\ell$ is composed with polynomials.
We want to characterize

$$\mathbb{P}^{[\mathbf{q}]} = \{\mathbf{p} \in \mathbb{R}^m | \exists \mathbf{q} \in [\mathbf{q}], \mathbf{f}(\mathbf{p}, \mathbf{q}) = \mathbf{0}\}$$

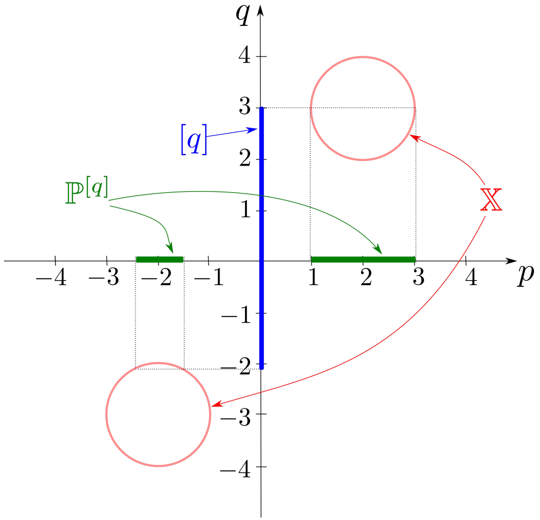
where $\mathbf{x} = (\mathbf{p}, \mathbf{q})$ and $[\mathbf{q}]$ is a box of $\mathbb{R}^\ell$.

Example.

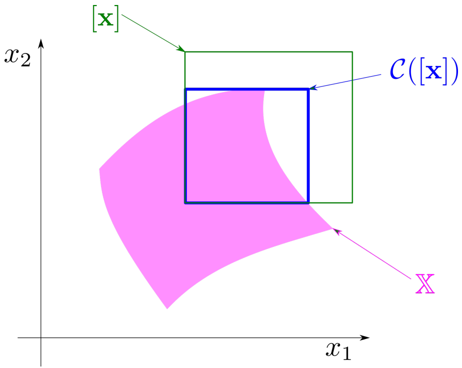
$$f(x_1,x_2) = x_1^4 + x_2^4 + 2x_1^2x_2^2 - 48x_1x_2 - 12x_2^2 + 8x_1^2 + 144$$

$$\mathbf{x} = (x_1,x_2) = (p,q).$$

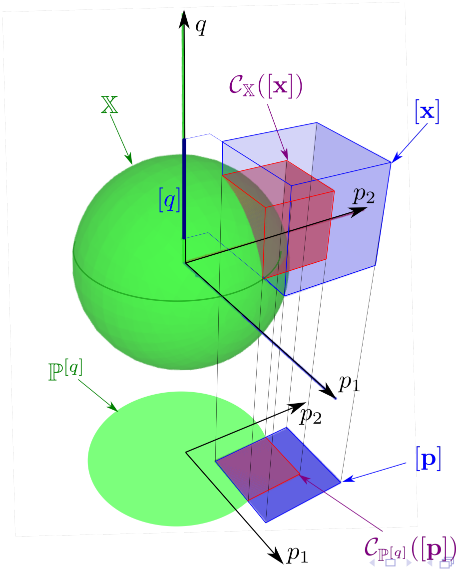
$$\mathbb{P}^{[q]} = \{p \in \mathbb{R} | \exists q \in [q], f(p,q) = 0\},$$


q
p
$[q]$
$\mathbb{P}^{[q]}$
$\mathbb{X}$
4
3
2
1
−1
−2
−3
−4
−4
−3
−2
−1
1
2
3
4

When $\mathbb{X}$ have symmetries, minimal contractors can be found [1]

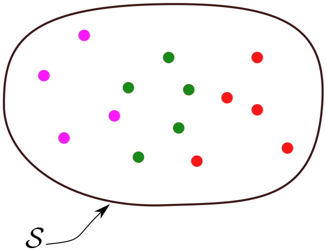


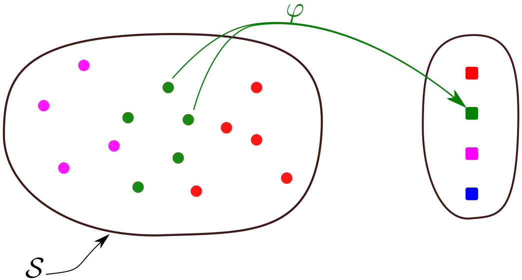
Now, minimality of contractors is preserved by projection ...

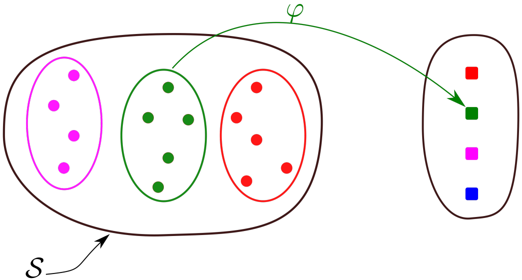

q
$\mathcal{C}_{\mathbb{X}}([\mathbf{x}])$
$[\mathbf{x}]$
$\mathbb{X}$
$[q]$
p_2
p_1
$\mathbb{P}^{[q]}$
p_2
$[\mathbf{p}]$
$\mathcal{C}_{\mathbb{P}^{[q]}}([\mathbf{p}])$
p_1

Theory

Quotient

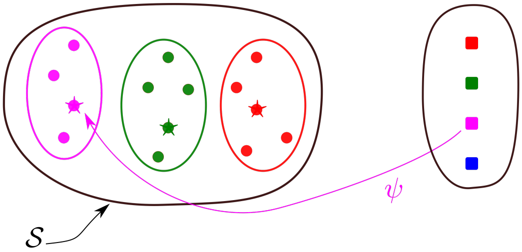

$\mathcal{S}$


φ
$\mathcal{S}$

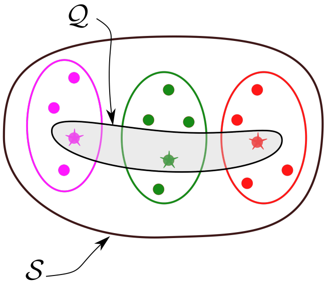


$$\sigma_1 \sim \sigma_2 \Leftrightarrow \varphi(\sigma_1) = \varphi(\sigma_2)$$

$\sim$ is the equivalence kernel of φ

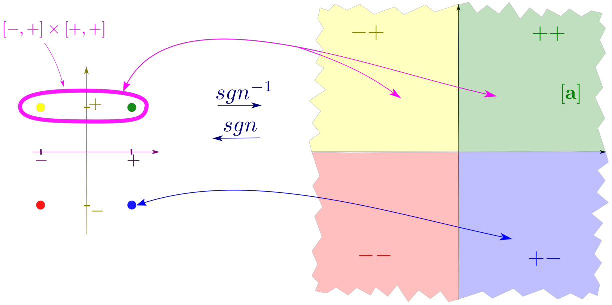


From the projection theorem, there exists a *choice function* ψ.
The images $\psi(\varepsilon)$ by ψ are the *representative.*



$$\mathscr{Q} = \frac{\mathscr{S}}{\sim}$$

Quadrants


$[-,+] \times [+,+]$
$+$
$-$
$-$
$+$
sgn^{-1}
sgn
$-+$
$++$
$[\mathbf{a}]$
$--$
$+-$

Interval extension of the sign

If $[\mathbf{q}] = [-1,2] \times [1,2] \times [-2,2]$, then

$$
\begin{array}{rcl}
sgn([\mathbf{q}]) & = & [-,+] \times [+,+] \times [-,+]] \\
 & = & \{(-+-),(-++),(++-]),(+++])\} \\
 & = & \left\{ \begin{pmatrix} \mathbb{R}^- \\ \mathbb{R}^+ \\ \mathbb{R}^- \end{pmatrix}, \begin{pmatrix} \mathbb{R}^- \\ \mathbb{R}^+ \\ \mathbb{R}^+ \end{pmatrix}, \begin{pmatrix} \mathbb{R}^+ \\ \mathbb{R}^+ \\ \mathbb{R}^- \end{pmatrix}, \begin{pmatrix} \mathbb{R}^+ \\ \mathbb{R}^+ \\ \mathbb{R}^+ \end{pmatrix} \right\}
\end{array}
$$

Action

Action of σ on the set $\mathbb{X}$ as

$$\sigma\mathbb{X} \quad = \quad \{\mathsf{y} \,|\, \sigma^{-1}(\mathsf{y}) \in \mathbb{X}\}$$

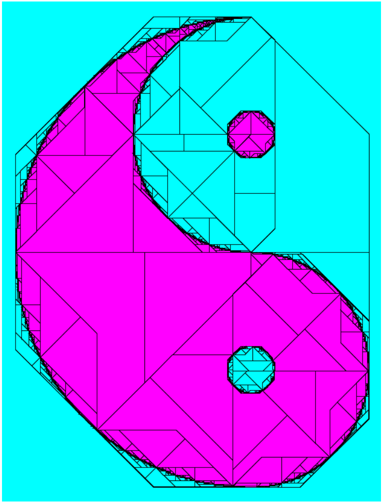
The set $\mathbb{X}$ is symmetric with respect to σ if $\sigma\mathbb{X} = \mathbb{X}$.

Moore family

$\mathbb{IM}$ is a Moore family of $\mathbb{R}^n$ if

$$[a](1) \in \mathbb{IM}, [a](2) \in \mathbb{IM}, \ldots \;\Rightarrow \bigcap_i [a](i) \in \mathbb{IM}$$

Contractors can be generalized [2] to Moore abstract domains.



Octogones (Damien Massé)

A group of symmetries $\Sigma = \{\sigma_1, \sigma_2, \dots\}$ is $\mathbb{IM}$-conservative if

$$\sigma \in \Sigma, [a] \in \mathbb{IM} \Rightarrow \sigma[a] \in \mathbb{IM}$$

For instance a rotation of $k\frac{\pi}{4}$ is octogone conservative.

A Moore family can (often) be generated by

- a generator (a subset of $\mathbb{R}^n$)
- a group of symmetries $\Sigma = \{\sigma_1, \sigma_2, \ldots\}$

For instance, boxes can be generated by

- the unit cube $[-1, 1]^n$
- scales and translations.

For instance, octogones can be generated by

- the unit cube $[-1,1]^n$
- scales, translations and rotations of $k\frac{\pi}{4}$.

Factorizability

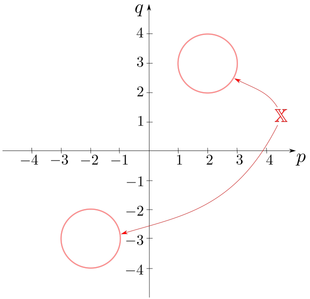
A set $\mathbb{X} \subset \mathbb{R}^n$ is Σ-factorizable if

$$\mathbb{X} = \sigma_1 \mathbb{X}_1 \cup \cdots \cup \sigma_k \mathbb{X}_k$$

where minimal contractors exist for the $\mathbb{X}_i$'s

Example. Take

$$\mathbb{X} = \left\{ \mathbf{x} \in \mathbb{R}^n | x_1^4 + x_2^4 + 2x_1^2 x_2^2 - 48x_1 x_2 - 12x_2^2 + 8x_1^2 + 144 = 0 \right\}$$



$$\mathbb{X}_1 = \left\{ \mathbf{x} \in \mathbb{R}^n | (x_1 - 2)^2 + (x_2 - 3)^2 - 1 = 0 \right\}$$

$$\sigma(x_1, x_2) = (-x_1, -x_2)$$

$$\sigma\mathbb{X}_1 = \left\{ \mathbf{x} \in \mathbb{R}^n | (x_1 + 2)^2 + (x_2 + 3)^2 - 1 = 0 \right\}$$

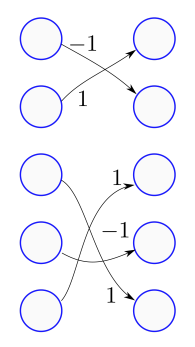
Factorization:

$$\mathbb{X} = \mathbb{X}_1 \cup \sigma\mathbb{X}_1$$

Hyperoctahedral symmetries

B_n which is the group of symmetries of the hypercube $[-1,1]^n$.
Example :

$$\sigma(x_1,x_2,x_3,x_4,x_5)=(-x_2,x_1,x_5,-x_4,x_3).$$



$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 1 & 5 & -4 & 3 \end{pmatrix}$$

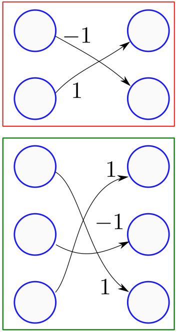
Multiplication :

$$\mathbf{u} \cdot \mathbf{v} = (\mathrm{sign}(v_1) \cdot u_{|v_1|}, \ldots, \mathrm{sign}(v_n) \cdot u_{|v_n|}).$$

Projection method

Separability

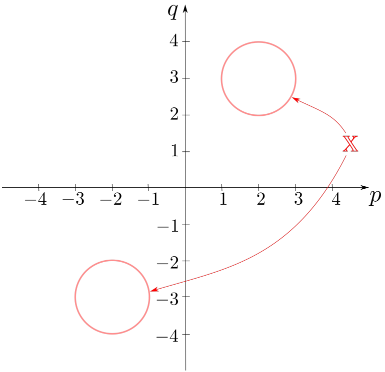
A symmetry $\sigma \in B_n$ is *separable* if $\sigma = (\sigma_{\mathbf{p}}, \sigma_{\mathbf{q}})$, where $\sigma_{\mathbf{p}}, \sigma_{\mathbf{q}}$ are symmetries in the **p**-space and the **q**-space respectively.



$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 1 & 5 & -4 & 3 \end{pmatrix}$$

The group of all $\sigma \in B_n$ that are symmetries for $\mathbb{X}$ is denoted by $B_n(\mathbb{X})$.



$$B_2(\mathbb{X}) = \{(1,2),(-1,-2)\}.$$

Quotient

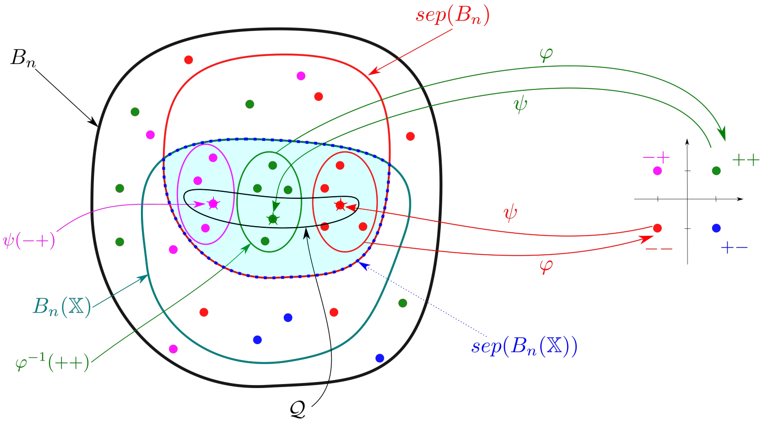
Define $\varphi(\sigma_{\mathbf{p}}, \sigma_{\mathbf{q}}) = \mathrm{sign}(\sigma_{\mathbf{q}})$. For instance, for $dim\, \mathbf{p} = 2$,

$$\varphi(2,1,-4,3,6,-5) = \mathrm{sign}(-4,3,6,-5) = (-++-).$$

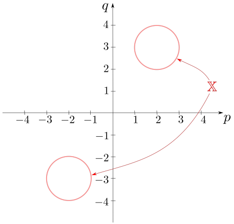
Define

$$\mathscr{Q} = \frac{sep(B_n(\mathbb{X}))}{\sim}.$$

where $\sim$ is the equivalence kernel of φ.
$\mathscr{Q}$ gives us the symmetries to move from one $\mathbf{q}-$quadrant to the positive $\mathbf{q}$-quadrant.



Choice function ψ



For our two-disks example:

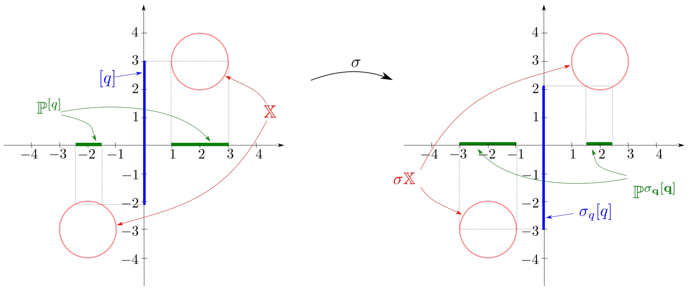
$$
\begin{aligned}
B_2 &= \{(-1,-2),(-1,2),\ldots,(2,-1),(2,1)\} \\
sep(B_2) &= \{(-1,-2),(-1,2),(1,-2),(1,2)\} \\
sep(B_2(\mathbb{X})) &= \{(-1,-2),(1,2)\} \\
\mathscr{Q} &= \{(-1,-2),(1,2)\} \\
\psi(-) &= (-1,-2) \\
\psi(+) &= (1,2)
\end{aligned}
$$

Proposition. Consider $\sigma = \sigma_{\mathbf{p}} \times \sigma_{\mathbf{q}} \in sep(B_n(\mathbb{X}))$ and

$$\mathbb{P}^{[\mathbf{q}]} = \{\mathbf{p} \in \mathbb{R}^m | \exists \mathbf{q} \in [\mathbf{q}], (\mathbf{p}, \mathbf{q}) \in \mathbb{X}\}$$

We have

$$\mathbb{P}^{[\mathbf{q}]} = \sigma_{\mathbf{p}} \mathbb{P}^{\sigma_{\mathbf{q}}[\mathbf{q}]}.$$


$[q]$
$\mathbb{P}^{[q]}$
$\mathbb{X}$
σ
$\sigma\mathbb{X}$
$\mathbb{P}^{\sigma_\mathbf{q}[\mathbf{q}]}$
$\sigma_q[q]$

$$\sigma = (-1, -2)$$

Theorem. Assume that $dom(\psi) = [-,+]^{\ell}$. We have

$$\mathbb{P}^{[\mathbf{q}]} = \bigcup_{\sigma \in \psi(sign([\mathbf{q}]))} \boldsymbol{\sigma}_{\mathbf{p}} \mathbb{P}^{(\mathbb{R}^{+\ell}) \cap \boldsymbol{\sigma}_{\mathbf{q}}[\mathbf{q}]}$$

Consequence. From a contractor working for $[\mathbf{q}] \subset \mathbb{R}^{n+}$, if we have enough symmetries, we have a contractor for any $[\mathbf{q}] \subset \mathbb{R}^{n}$

Compute $sep(B_n(\mathbb{X}))$

$$\begin{aligned} \mathbf{x} \in \mathbb{X}, \sigma(\mathbf{x}) \notin \mathbb{X} &\quad \Rightarrow \quad \sigma \notin B_n(\mathbb{X}) \\ \sigma_1 \in B_n(\mathbb{X}), \sigma_2 \in B_n(\mathbb{X}) &\quad \Rightarrow \quad \sigma_1 \sigma_2 \in B_n(\mathbb{X}) \end{aligned}$$

Input: $\mathbf{f}(\mathbf{p},\mathbf{q})$, $\mathscr{X} = \{\mathbf{x}_1, \mathbf{x}_2, \dots\}$, $\mathscr{S} = \{\sigma_1, \sigma_2, \dots\}$

$m = \dim \mathbf{p}$; $\ell = \dim \mathbf{q}$; $n = m + \ell$
Generate the list $\mathscr{B}$ of all symmetries in B_n
Remove from $\mathscr{B}$ all σ that are not separable
Remove from $\mathscr{B}$ all σ such that $\sigma(\mathbf{x}_i) \notin \mathbb{X}$ for some $\mathbf{x}_i \in \mathscr{X}$
Compute the group $\langle \mathscr{S} \rangle$ generated by $\mathscr{S}$
If $\mathscr{B} = \langle \mathscr{S} \rangle$ then return $\mathscr{B}$
return "Fail: add symmetries to $\mathscr{S}$ or solutions to $\mathscr{X}$".

Build the choice function ψ

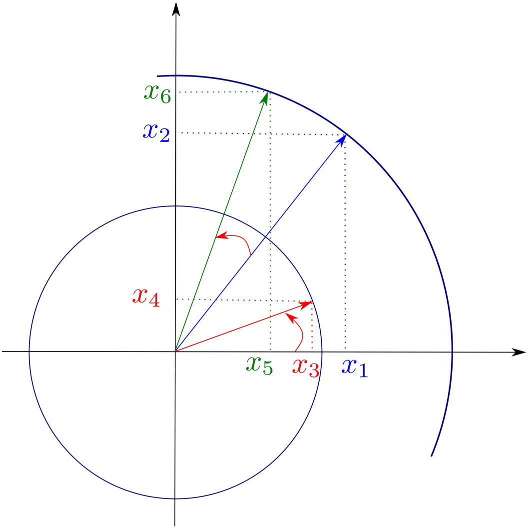
	Input: $\mathscr{Q}$
1	For each (σ_1, σ_2) of $\mathscr{S}$ such that $\varphi(\sigma_1) = \varphi(\sigma_2)$, remove σ_2 from $\mathscr{Q}$.
2	For each $\varepsilon \in [-,+]^\ell$, define $\psi(\varepsilon) = \varphi^{-1}_{\|\mathscr{Q}}$.
3	return ψ

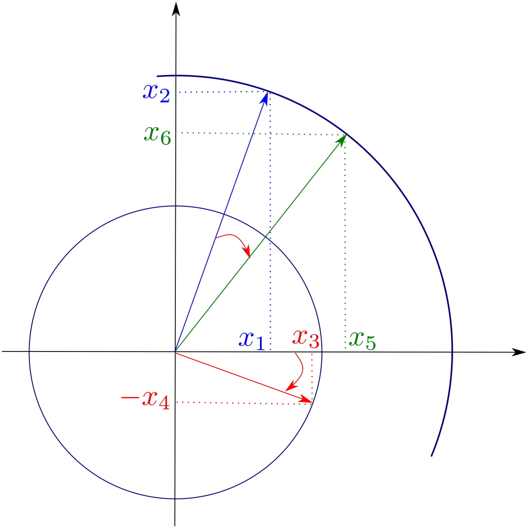
Rotate

Consider the *rotate* constraint

$$\left\{\begin{array}{ccc} x_3x_1 - x_4x_2 - x_5 & = & 0 \\ x_4x_1 + x_3x_2 - x_6 & = & 0 \\ x_3^2 + x_4^2 - 1 & = & 0 \end{array}\right.$$

$$\left\{\begin{array}{rcl} x_3x_1 - x_4x_2 - x_5 & = & 0 \\ x_4x_1 + x_3x_2 - x_6 & = & 0 \\ x_3^2 + x_4^2 - 1 & = & 0 \end{array}\right. \Leftrightarrow \left\{\begin{array}{rcl} x_3x_5 + x_4x_6 - x_1 & = & 0 \\ -x_4x_5 + x_3x_6 - x_2 & = & 0 \\ x_3^2 + (-x_4)^2 - 1 & = & 0 \end{array}\right.$$


x_6
x_2
x_4
x_5
x_3
x_1


x_2
x_6
x_1
x_3
x_5
$-x_4$

Consider the *rotate* constraint:

$$\begin{cases} q_1 p_1 - q_2 p_2 & = & q_4 \\ q_2 p_1 + q_1 p_2 & = & q_5 \\ q_1^2 + q_2^2 - 1 & = & 0 \end{cases}$$

We get 46080 elements for B_6.
To find $B_6(\mathbb{X})$, we took

$$\mathscr{S} = \{(2,-1,-4,3,5,6),(1,-2,3,-4,5,-6),(-1,-2,-4,3,6,-5)\}.$$

We also provided an element of $\mathbb{X}$

$$\mathbf{x}_1 = \left(2, 4, \frac{1}{2}, \frac{\sqrt{3}}{2}, 1-2\sqrt{3}, \sqrt{3}+2\right).$$

We get a list of 32 elements for the group $sep(B_6(\mathbb{X}))$.

$$
\psi : \begin{cases}
(++++) & \mapsto & (1,2,3,4,5,6) \\
(+++-) & \mapsto & (2,-1,3,4,6,-5) \\
(++-+) & \mapsto & (-2,1,3,4,-6,5) \\
(++--) & \mapsto & (-1,-2,3,4,-5,-6) \\
(+-++) & \mapsto & (2,1,3,-4,6,5) \\
(+-+-) & \mapsto & (1,-2,3,-4,5,-6) \\
\color{blue}{(+--+)} & \color{blue}{\mapsto} & \color{blue}{(-1,2,3,-4,-5,6)} \\
(+---) & \mapsto & (-2,-1,3,-4,-6,-5) \\
(-+++) & \mapsto & (-2,-1,-3,4,6,5) \\
(-++-) & \mapsto & (-1,2,-3,4,5,-6) \\
(-+-+) & \mapsto & (1,-2,-3,4,-5,6) \\
(-+--) & \mapsto & (2,1,-3,4,-6,-5) \\
(--++) & \mapsto & (-1,-2,-3,-4,5,6) \\
(--+-) & \mapsto & (-2,1,-3,-4,6,-5) \\
(---+) & \mapsto & (2,-1,-3,-4,-6,5) \\
(----) & \mapsto & (1,2,-3,-4,-5,-6)
\end{cases}
$$

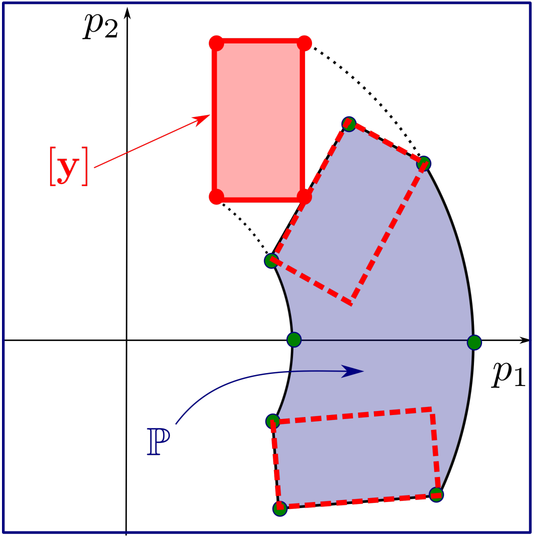
Note that $\sigma = \psi(+--+) = (-1, 2, 3, -4, -5, 6)$, corresponds to:

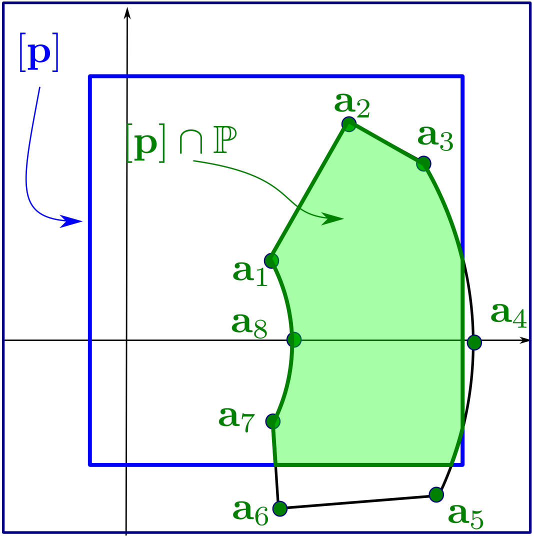
$$\begin{pmatrix} q_1 p_1 - q_2 p_2 - q_4 \\ q_2 p_1 + q_1 p_2 - q_5 \\ q_1^2 + q_2^2 - 1 \end{pmatrix} = \mathbf{0} \Leftrightarrow \begin{pmatrix} q_1(-p_1) - (-q_2)p_2 - (-q_4) \\ (-q_2)(-p_1) + q_1 p_2 - q_5 \\ q_1^2 + (-q_2)^2 - 1 \end{pmatrix} = \mathbf{0}$$

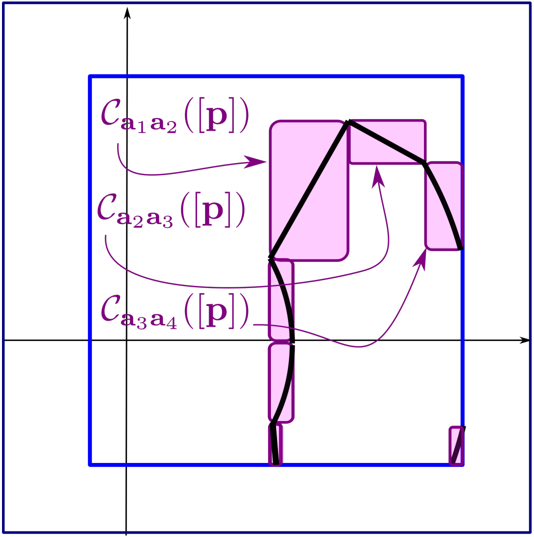
σ transports the quadrant $(+--+)$ of the $\mathbf{q}$-space to $(++++)$.

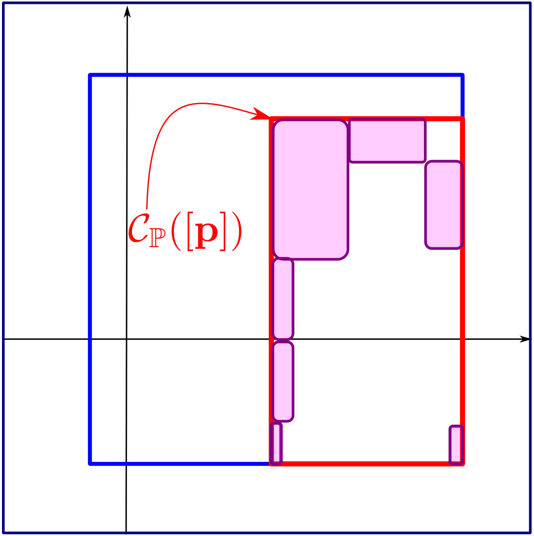
Positive quadrant

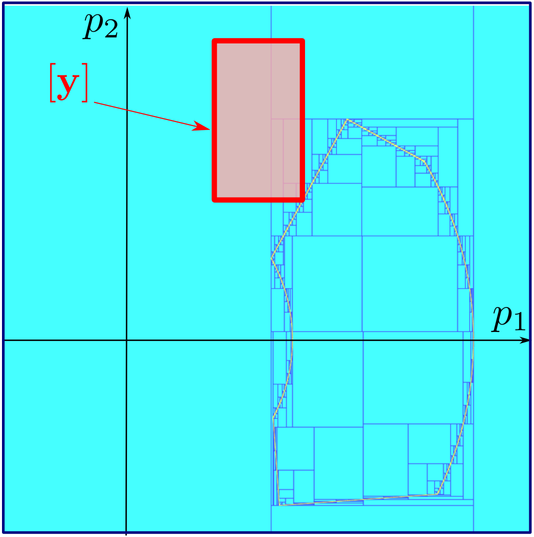
- We use a boundary approach
- We take advantage of the monotonicity
- The union of minimal contractors is minimal

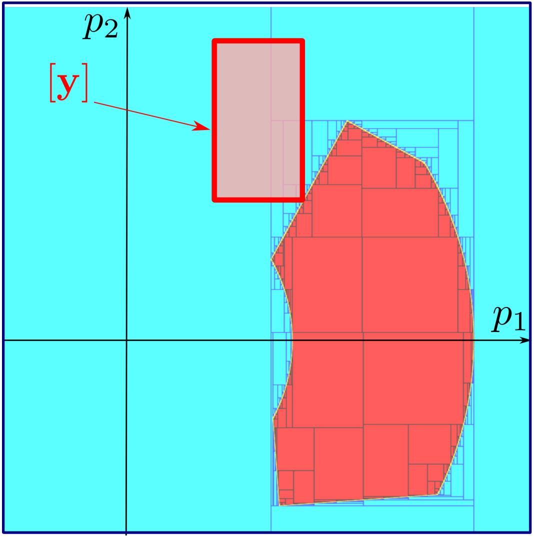

p_2
$[\mathbf{y}]$
p_1
$\mathbb{P}$


$[\mathbf{p}]$
$[\mathbf{p}] \cap \mathbb{P}$
$\mathbf{a}_2$
$\mathbf{a}_3$
$\mathbf{a}_1$
$\mathbf{a}_8$
$\mathbf{a}_4$
$\mathbf{a}_7$
$\mathbf{a}_6$
$\mathbf{a}_5$


$\mathcal{C}_{\mathbf{a}_1\mathbf{a}_2}([\mathbf{p}])$
$\mathcal{C}_{\mathbf{a}_2\mathbf{a}_3}([\mathbf{p}])$
$\mathcal{C}_{\mathbf{a}_3\mathbf{a}_4}([\mathbf{p}])$


$\mathcal{C}_{\mathbb{P}}([\mathbf{p}])$


p_2
$[\mathbf{y}]$
p_1


p_2
$[\mathbf{y}]$
p_1

All quadrants

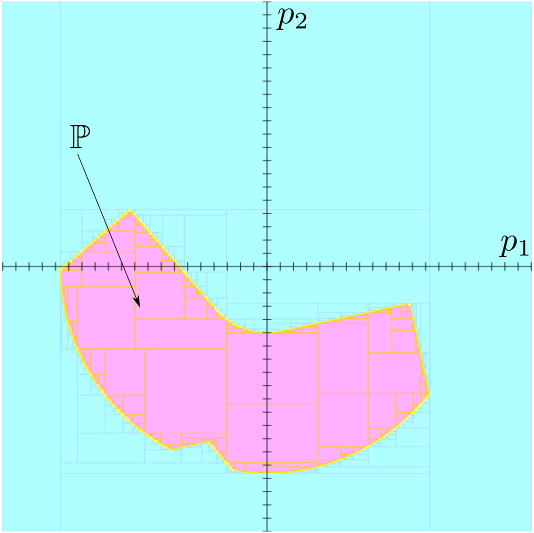
Take

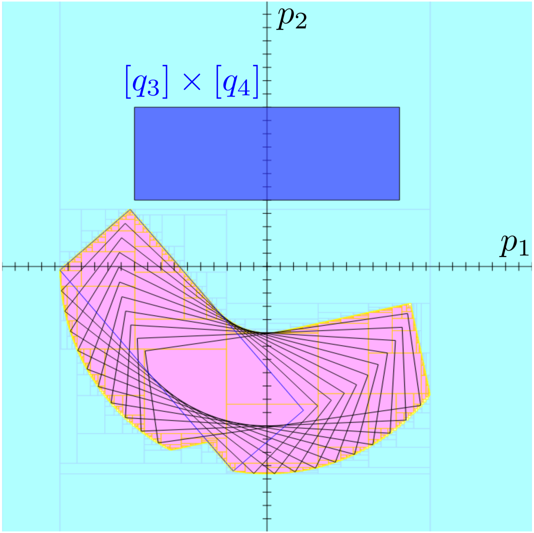
$$[\mathbf{q}] = (\cos([\theta]), \sin([\theta]), [-10,10], [5,12])\,.$$

with $[\theta] = [3,4]$.

We have

$$\mathbb{P}^{[\mathbf{q}]} = \bigcup_{\sigma \in \psi(sign([\mathbf{q}]))} \boldsymbol{\sigma}_{\mathbf{p}} \mathbb{P}^{\mathbb{R}^{+4} \cap \sigma_{\mathbf{q}}[\mathbf{q}]}.$$


p_2
p_1
$\mathbb{P}$


p_2
$[q_3] \times [q_4]$
p_1

Applications

Workspace

We consider a polygon $\mathscr{M}$ with coordinates

$$\begin{pmatrix} 5 & 5 & -4 & -4 & 11 & 11 & 10 & 10 \\ -8 & 8 & 8 & 10 & 10 & 8 & 8 & -8 \end{pmatrix}$$

i.e.,

$$\mathscr{M} = \underbrace{([5,10] \times [-8,8])}_{=[\mathbf{m}](1)} \cup \underbrace{([-4,11] \times [8,10])}_{=[\mathbf{m}](2)}.$$

The object can rotates around $\mathbf{0}$ with an angle $\theta \in [\theta] = [0.5, 1.5]$. The workspace [4] corresponds to all space that can occupy the object.

Define

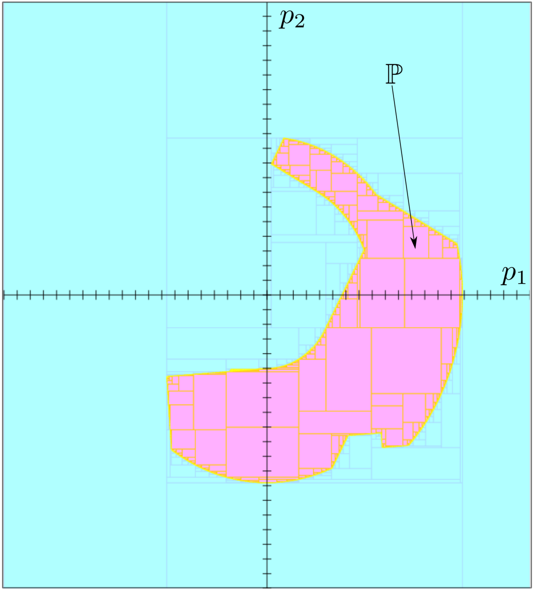
i	1	2
$[q_1](i)$	$\cos([0.5, 1.5])$	$\cos([0.5, 1.5])$
$[q_2](i)$	$\sin([0.5, 1.5])$	$\sin([0.5, 1.5])$
$[q_3](i)$	$[5, 10]$	$[-8, 8]$
$[q_4](i)$	$[-4, 11]$	$[8, 10]$

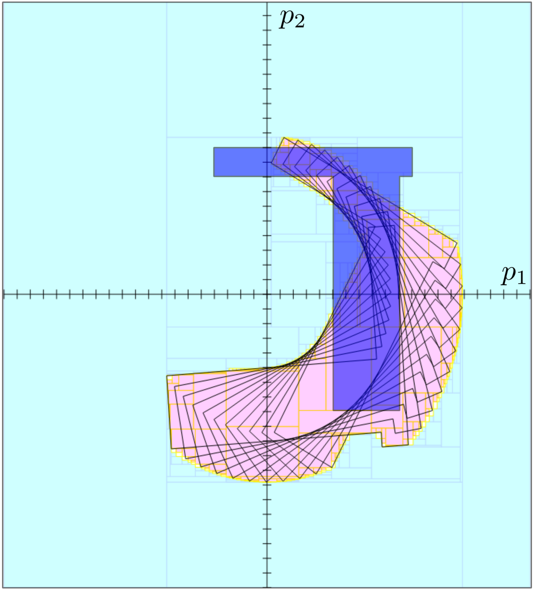
The workspace is

$$\mathbb{P} \;=\; \bigcup_{i \in \{1,2\}} \mathbb{P}_i$$

where

$$\mathbb{P}_i = \left\{ \mathbf{p} \in \mathbb{R}^2 \,|\, \exists \mathbf{q} \in [\mathbf{q}](i),\, rotate(\mathbf{p}, \mathbf{q}) \right\}$$


p_2
$\mathbb{P}$
p_1


p_2
p_1

Speed estimation

An object moves with an unknown speed $\mathbf{v}$. The speed is measured by 6 robots.
In their own frame, each robot returns a box $[\mathbf{y}]$ enclosing the speed vector.
The heading of the ith robot θ_i is badly known.

i	1	2	3	4	5	6
θ_i	$[1,2]$	$[2,3]$	$[3,4]$	$[-0,1]$	$[-2,-1]$	$[-3,-2]$
y_1^i	13 ± 1	-1 ± 1	-9 ± 1	11 ± 1	-7 ± 1	-13 ± 1
y_2^i	-5 ± 1	-15 ± 1	-11 ± 1	9 ± 1	11 ± 1	3 ± 1

The set of all feasible speed vectors is

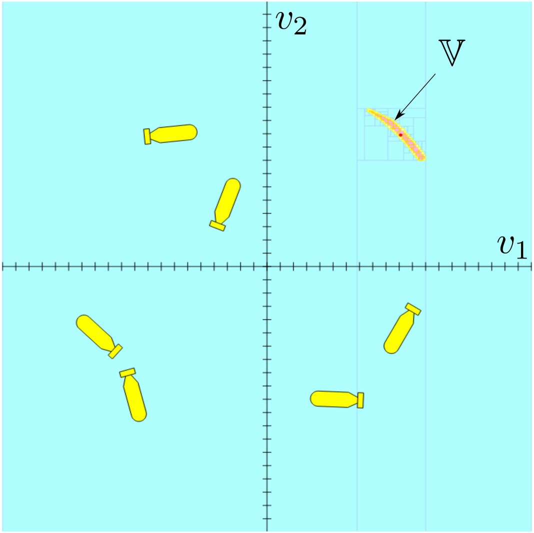
$$\mathbb{V} = \bigcap_{i\in\{1,\ldots,6\}} \mathbb{V}_i$$

where

$$\mathbb{V}_i = \left\{\mathbf{v} \in \mathbb{R}^2 \,|\, \exists \mathbf{q} \in [\mathbf{q}](i), \mathit{rotate}(\mathbf{v},\mathbf{q})\right\}$$

and

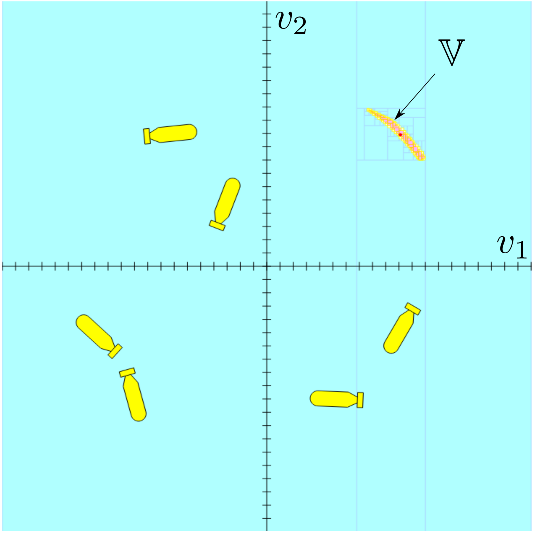
$$[\mathbf{q}](i) = \cos([\theta^i]) \times \sin([\theta^i]) \times [y_1^i] \times [y_2^i]$$

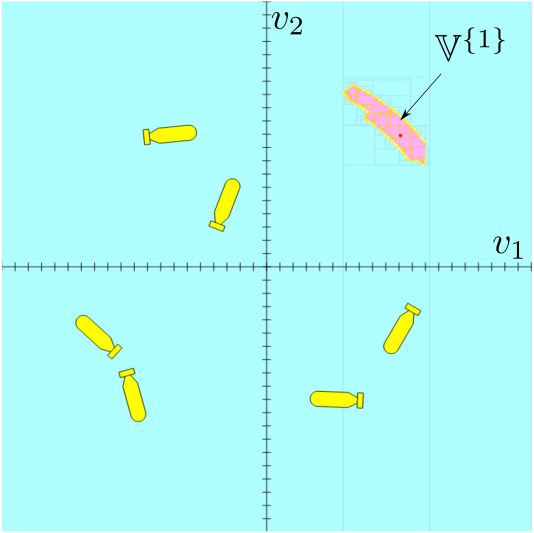


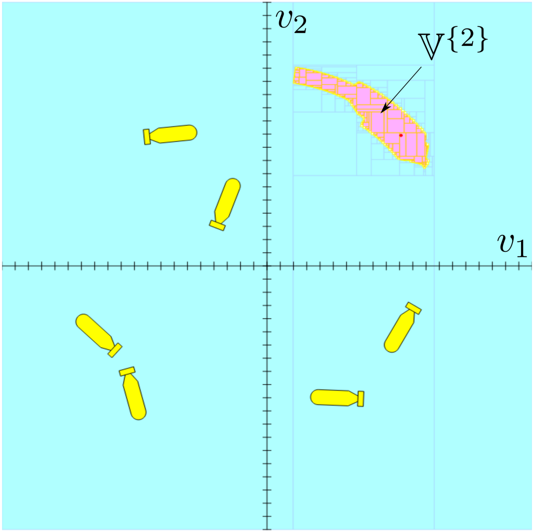
Estimation of the speed vector

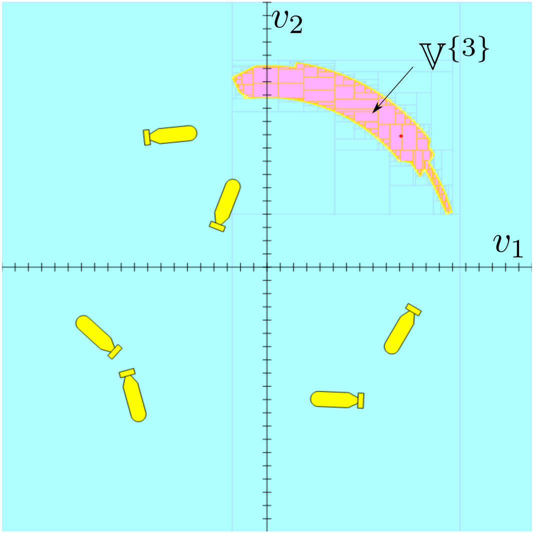
Define the relaxed intersection [3] as

$$\mathbb{V}^{\{k\}} \quad = \quad \bigcap_{i\in\{1,\dots,6\}}^{\{k\}} \mathbb{V}_i \ .$$


v_2
$\mathbb{V}$
v_1


v_2
$\mathbb{V}^{\{1\}}$
v_1


v_2
$\mathbb{V}^{\{2\}}$
v_1


v_2
$\mathbb{V}^{\{3\}}$
v_1

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