Guaranteed approximations of arbitrarily quantified reachability problems

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Motivation

Robust reachability - given $\phi(t; x_0, u, v)$ the flow of an ODE at time t from x_0 with control u and disturbance w

For time $t \in [0, T]$, compute:

$$R_{\forall\exists}(\varphi)(t) = \{z \mid \forall w \in \mathbb{W}, \ \exists x_0 \in \mathbb{X}_0, \ \exists u \in \mathbb{U}, z = \varphi(t; x_0, u, w)\}$$

(can a controller compensate disturbances or change of values of parameters that are known to the controller?)

"Even more" robust (but needs some time and/or space relaxation)

Can a controller not knowing the disturbance still reach the target, up to some (time) relaxation?

$$R_{\exists\forall\exists}(\varphi) = \{z \in \mathbb{R}^m \mid \exists u \in \mathbb{U}, \ \exists x_0 \in \mathbb{X}_0, \ \forall w \in \mathbb{W}, \ \exists s \in [0, T], \ z = \varphi(s; x_0, u, w)\}$$

But also

Motion planning

Go through regions S_j between times T_{j-1} and T_j, $j = 1, \ldots, k$, final states z_k?

$$\{z_k \in \mathbb{R}^m \mid \exists u_1 \in \mathbb{U},\ \forall x_0 \in \mathbb{X}_0,\ \forall w_1 \in \mathbb{W},\ \exists t_1 \in [0, T_1],\ \exists z_1 \in S_1$$
$$\exists u_2 \in \mathbb{U},\ \forall w_2 \in \mathbb{W},\ \exists t_2 \in [T_1, T_2],\ \exists z_2 \in S_2,\ \ldots$$
$$\exists u_k \in \mathbb{U},\ \forall w_k \in \mathbb{W},\ \exists t_k \in [T_{k-1}, T],$$
$$\begin{pmatrix} z_1 \\ z_2 \\ \ldots \\ z_k \end{pmatrix} = \begin{pmatrix} \varphi(t_1; u_1, x_0, w_1) \\ \varphi(t_2 - t_1; u_2, z_1, w_2) \\ \ldots \\ \varphi(t_k - t_{k_1}; u_k, z_{k-1}, w_k) \end{pmatrix}$$

General temporal logics formulas, and hyperproperties

E.g. behavioral robustness, or comparisons of controllers:

$$R_{\exists\forall\exists\forall\exists}(\varphi) = \{z \mid \exists x_0 \in \mathbb{X}_0,\ \exists \delta \in [-\epsilon, \epsilon]^i,$$
$$\forall u \in \mathbb{U},\ \exists u' \in \mathbb{U},\ \forall w \in \mathbb{W},\ \exists t \in [T_1, T_2],$$
$$z = \|\varphi(t; x_0, u, w) - \varphi(t; x_0 + \delta, u', w)\|\}$$

Problem statement

Notations

- $f : \mathbb{R}^p \to \mathbb{R}^m$ (e.g. flow function etc.)
- the p arguments of f partitioned into consecutive j_i arguments $i = 1, \ldots, 2n$ corresponding to the alternations of quantifiers, with $p = \sum_{i=1}^{2n} j_i$.
- partition identified with sequence $(j_1, \ldots, j_{2n})$, denoted by $\boldsymbol{p}$.
- we note: $\boldsymbol{x}_i = (x_{k_i+1}, \ldots, x_{k_{i+1}})$ where $k_i = \sum_{l=1}^{i-1} j_l$, $i = 1, \ldots, 2n+1$, and

$$f(x_1, x_2, \ldots, x_{k_{2n}}) = f(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_{2n})$$

General quantified problems

n alternations of quantifiers $\forall\exists$ reachability problem:

$$R_{\boldsymbol{p}}(f) = \Big\{ z \in \mathbb{R}^m \mid \forall \boldsymbol{x}_1 \in [-1,1]^{j_1},\ \exists \boldsymbol{x}_2 \in [-1,1]^{j_2},\ \ldots,$$
$$\forall \boldsymbol{x}_{2n-1} \in [-1,1]^{j_{2n-1}}, \exists \boldsymbol{x}_{2\boldsymbol{n}} \in [-1,1]^{j_{2n}},\ z = f(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{2n}) \Big\}$$

On the generality of these quantified problems

Remarks

- Add dummy existential quantifier (resp. universal quantifier) at the beginning (resp. end) for getting all quantified formulas
- Up to reparametrization, quantified problems with other boxes than $[-1,1]^{j_i}$
- Also possible to consider more general sets over which to quantify variables x_i by suitable outer and inner approximations as boxes
- Can consider e.g. control u and disturbance w as piecewise constant signals over a bounded time horizon.

Example

Dubbins vehicle

$$\begin{pmatrix} \dot{x} \\ \dot{y} \\ \dot{\theta} \end{pmatrix} = \begin{pmatrix} v cos(\theta) + b_1 \\ v sin(\theta) \\ a \end{pmatrix}$$

- Control period of $t = 0.5$, linear velocity $v = 1$,
- Initial conditions: $\mathbb{X}_0 = \{(x, y, \theta) \mid x \in [-0.1, 0.1],\ y \in [-0.1, 0.1],\ \theta \in [-0.01, 0.01]\}$,
- Control a (angular velocity) in $\mathbb{U} = [-0.01, 0.01]$,
- disturbance b_1 in $\mathbb{W} = [-0.01, 0.01]$

We want to estimate:

$$R_{\exists\forall\exists}(\varphi) = \{z \in \mathbb{R}^m \mid \exists u \in \mathbb{U},\ \exists x_0 \in \mathbb{X}_0,\ \forall w \in \mathbb{W},\ \exists s \in [0, T],\ z = \varphi(s; x_0, u, w)\}$$

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First step: scalar affine functions

Notations

f is the affine function:

$$f(\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_{2n}) = \delta_0 + \langle \Delta_1, \boldsymbol{x}_1 \rangle + \langle \Delta_2, \boldsymbol{x}_2 \rangle + \ldots + \langle \Delta_{2n}, \boldsymbol{x}_{2n} \rangle$$

with $\Delta_i = (\delta_{k_i+1}, \ldots, \delta_{k_{i+1}}) \in \mathbb{R}^{j_i}$, $i = 1, \ldots, 2n$, where $k_i = \sum_{l=1}^{i-1} j_l$.

Exact characterization

$$R_{\boldsymbol{p}}(f) = \delta_0 + \left[\sum_{k=1}^{n} (||\Delta_{2k-1}|| - ||\Delta_{2k}||) \, , \, \sum_{k=1}^{n} (||\Delta_{2k}|| - ||\Delta_{2k-1}||) \right]$$

if $||\Delta_{2l-1}|| \leq ||\Delta_{2l}|| + \sum_{k=l+1}^{n} (||\Delta_{2k}|| - ||\Delta_{2k-1}||)$ for $l = 1, \ldots, n$, otherwise $R_{\boldsymbol{p}}(f) = \emptyset$

The non-vacuity condition is paramount

Notations

Function f from $\mathbb{R}^2$ to $\mathbb{R}$, consider:

$$\begin{aligned} R_{\forall\exists}(f) &= \{z \mid \forall x_2,\ \exists x_1,\ z = f(x_1, x_2)\} \\ R_{\exists\forall}(f) &= \{z \mid \exists x_1,\ \forall x_2,\ z = f(x_1, x_2)\} \end{aligned}$$

Difference between $\forall, \exists$ and $\exists, \forall$

We always have $R_{\exists\forall}(f) \subseteq R_{\forall\exists}(f)$, but, for any affine function $f(x_1, x_2) = a + bx_1 + cx_2$:

- If $c \neq 0$, $R_{\exists\forall}(f) = \emptyset$
- If $c = 0$, $R_{\exists\forall}(f) = [a - |b|, a + |b|] = R_{\forall\exists}(f)$,

The case of non-linear scalar functions

Notations

- Function $f:\ \mathbb{R}^p \to \mathbb{R}$, $\boldsymbol{p} = (j_1, \ldots, j_{2n})$ partition of the p arguments of f, $k_l = \sum_{i=1}^{l-1} j_i$, for $l = 1, \ldots, 2n+1$.
- Suppose we have p intervals $A_1, \ldots, A_p$, write $\boldsymbol{A}_i = (A_{k_i+1}, \ldots, A_{k_{i+1}}),\ i = 1, \ldots, 2n$ for the corresponding boxes in $\mathbb{R}^{j_i}$,
- Consider the set:

$$\mathcal{C}(\boldsymbol{A}_1, \ldots, \boldsymbol{A}_{2n}) = \{z \mid \forall \alpha_1 \in \boldsymbol{A}_1,\ \exists \alpha_2 \in \boldsymbol{A}_2, \ldots, \forall \alpha_{2n-1} \in \boldsymbol{A}_{2n-1},\ \exists \alpha_{2n} \in \boldsymbol{A}_{2n},\ z = \sum_{j=1}^{2n} \alpha_j\}.$$

- And functions, for $j = 1, \ldots, p$:

$$h^{x_1,\ldots,x_{j-1}}(x_j) = f(x_1, \ldots, x_{j-1}, x_j, 0, \ldots, 0) - f(x_1, \ldots, x_{j-1}, 0, \ldots, 0)$$

The case of non-linear scalar functions

Characterization of $R_p(f)$ through linearizations

Given inner and outer-approximations of the images of functions $h^{x_1,\ldots,x_{j-1}}$, for $j = 1, \ldots, p$. :

$$I_j \subseteq range(h^{x_1,\ldots,x_{j-1}}) \subseteq O_j$$

Then, writing $\boldsymbol{I}_i = \prod_{j=k_i+1}^{k_{i+1}} [\underline{I}_j, \overline{I}_j]$, $\boldsymbol{O}_i = \prod_{j=k_i+1}^{k_{i+1}} [\underline{O}_j, \overline{O}_j]$, $i = 1, \ldots, 2n$:

$$f(0,\ldots,0) + \mathcal{C}(\boldsymbol{O}_1, \boldsymbol{I}_2, \ldots, \boldsymbol{O}_{2n-1}, \boldsymbol{I}_{2n}) \subseteq R_{\boldsymbol{p}}(f) \subseteq f(0,\ldots,0) + C(\boldsymbol{I}_1, \boldsymbol{O}_2, \ldots, \boldsymbol{I}_{2n-1}, \boldsymbol{O}_{2n})$$

How do we find simple inner and outer-approximations of functions?

Generalized mean-value theorem

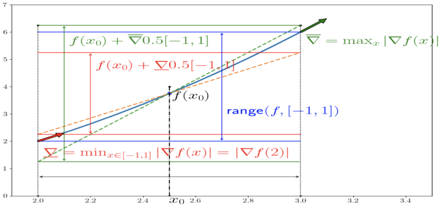
If we have, for all $i = 1, \ldots, 2n$ and all $j = k_i + 1, \ldots, k_{i+1}$, $\nabla_j = [\underline{\nabla}_j, \overline{\nabla}_j]$ such that:

$$\left\{ \left| \frac{\partial f}{\partial x_j}(\boldsymbol{x}_1, \ldots, \boldsymbol{x}_i, 0, \ldots, 0) \right| \mid \boldsymbol{x}_l \in [-1,1]^{j_l},\ l = 1, \ldots, i \right\} \subseteq \nabla_j$$

then, for all $j = 1, \ldots, 2n$:

$$I_i = \underline{\nabla}_j[-1,1], \quad O_j = \overline{\nabla}_j[-1,1]$$

give inner and outer-approximations of $range(h^{x_1,\ldots,x_{j-1}})$



(other approximation methods, higher-order in particular, see e.g. Eric Goubault Sylvie Putot, "Tractable higher-order under-approximating AE extensions for non-linear systems" ADHS 2021)

Finally

General formula for scalar, general functions

$$f(0,\ldots,0) + \left[\sum_{k=1}^{n} \sum \left(\overline{\boldsymbol{O}}_{2k-1} + \underline{\boldsymbol{I}}_{2k}\right),\ \sum_{k=1}^{n} \sum \left(\overline{\boldsymbol{I}}_{2k} + \underline{\boldsymbol{O}}_{2k-1}\right)\right] \subseteq R_{\boldsymbol{p}}(f)$$

if $\sum \overline{\boldsymbol{O}}_{2l-1} - \sum \underline{\boldsymbol{O}}_{2l-1} \leq \sum_{k=l}^{n} \sum \left(\overline{\boldsymbol{I}}_{2k} - \underline{\boldsymbol{I}}_{2k}\right) - \sum_{k=l+1}^{n} \sum \left(\overline{\boldsymbol{O}}_{2k-1} - \underline{\boldsymbol{O}}_{2k-1}\right)$ for $l = 1,\ldots,n$, otherwise the inner-approximation is empty, and:

$$R_{\boldsymbol{p}}(f) \subseteq f(0,\ldots,0) + \left[\sum_{k=1}^{n} \sum \left(\overline{\boldsymbol{I}}_{2k-1} + \underline{\boldsymbol{O}}_{2k}\right),\ \sum_{k=1}^{n} \sum \left(\overline{\boldsymbol{O}}_{2k} + \underline{\boldsymbol{I}}_{2k-1}\right)\right]$$

if $\sum \overline{\boldsymbol{I}}_{2l-1} - \sum \underline{\boldsymbol{I}}_{2l-1} \leq \sum_{k=l}^{n} \sum \left(\overline{\boldsymbol{O}}_{2k} - \underline{\boldsymbol{O}}_{2k}\right) - \sum_{k=l+1}^{n} \sum \left(\overline{\boldsymbol{I}}_{2k-1} - \underline{\boldsymbol{I}}_{2k-1}\right)$ for $l = 1,\ldots,n$, otherwise the outer-approximation is empty.

Looks a bit intimidating...

Example, function $g : \mathbb{R}^3 \to \mathbb{R}$ on $[-1,1]^3$

$$g(x_1, x_2, x_3) = \frac{x_1^2}{4} + (x_2+1)(x_3+2) + (x_3+3)^2.$$

Compute $R_{\exists\forall\exists}(g) = \{z \mid \exists x_1 \in [-1,1],\ \forall x_2 \in [-1,1],\ \exists x_3 \in [-1,1], z = g(x_1,x_2,x_3)\}$

"Individual contributions" of each argument

- $\nabla_1 = |\frac{\partial g}{\partial x_1}| = |\frac{x_1}{2}| \in \left[0, \frac{1}{2}\right]$, $\nabla_2 = |\frac{\partial g}{\partial x_2}| = |x_3+2| \in [1,3]$, $\nabla_3 = |\frac{\partial g}{\partial x_3}| = |x_2 + 1 + 2(x_3+3)| \in [4,10]$, and $c = g(0,0,0) = 11$.
- Therefore, outer and inner approximations: $O_1 = \left[-\frac{1}{2}, \frac{1}{2}\right]$, $I_1 = 0$, $O_2 = [-3,3]$, $I_2 = [-1,1]$ and $O_3 = [-10,10]$, $I_3 = [-4,4]$.

Outer-approximation of $R_{\exists\forall\exists}(g)$

$$\begin{array}{rccccccccl} [& c & +\underline{O}_1 & +\overline{I}_2 & +\underline{O}_3, & c & +\overline{O}_1 & +\underline{I}_2 & +\overline{O}_3 &] \\ = [& 11 & -\frac{1}{2} & +1 & -10, & 11 & +\frac{1}{2} & -1 & +10 &] = [1.5, 20.5] \end{array}$$

(in comparison, the sampling based estimation is $[6.25, 16.25]$)

Looks a bit intimidating...

Example, function $g : \mathbb{R}^3 \to \mathbb{R}$ on $[-1,1]^3$

Compute $R_{\exists\forall\exists}(g) = \{z \mid \exists x_1 \in [-1,1],\ \forall x_2 \in [-1,1],\ \exists x_3 \in [-1,1], z = g(x_1,x_2,x_3)\}$.

"Individual contributions" of each argument

- $\nabla_1 = |\frac{\partial g}{\partial x_1}| = |\frac{x_1}{2}| \in \left[0, \frac{1}{2}\right]$, $\nabla_2 = |\frac{\partial g}{\partial x_2}| = |x_3 + 2| \in [1,3]$, $\nabla_3 = |\frac{\partial g}{\partial x_3}| = |x_2 + 1 + 2(x_3+3)| \in [4,10]$, and $c = g(0,0,0) = 11$.
- Therefore, outer and inner approximations: $O_1 = \left[-\frac{1}{2}, \frac{1}{2}\right]$, $I_1 = 0$, $O_2 = [-3,3]$, $I_2 = [-1,1]$ and $O_3 = [-10,10]$, $I_3 = [-4,4]$.

Inner-approximation of $R_{\exists\forall\exists}(g)$

As $\overline{I}_3 + \underline{O}_2 = 1 \geq \underline{I}_3 + \overline{O}_2 = -1$:

$$\begin{array}{rcccccccccl} & [& c & +\underline{I}_1 & +\overline{O}_2 & +\underline{I}_3, & c & +\overline{I}_1 & +\underline{O}_2 & +\overline{I}_3 &] \\ = & [& 11 & 0 & +3 & -4, & 11 & +0 & -3 & +4 &] = [10,12] \end{array}$$

(in comparison, the sampling based estimation is $[6.25, 16.25]$)

Difference between $\forall\exists$ and $\exists\forall$ (II)

For function f from $\mathbb{R}^2$ to $\mathbb{R}$

$$\begin{aligned} R_{\forall\exists}(f) &= \{z \mid \forall x_2,\ \exists x_1,\ z = f(x_1, x_2)\} \\ R_{\exists\forall}(f) &= \{z \mid \exists x_1,\ \forall x_2,\ z = f(x_1, x_2)\} \end{aligned}$$

Recall, in any case, $R_{\exists\forall}(f) \subseteq R_{\forall\exists}(f)$

When f is non-linear, an example

- $f(x_1, x_2) = (x_1^2 - 1)x_2 + x_1$ for $x_1 \in [-1, 1]$ and $x_2 \in [-1, 1]$
- We have: $R_{\forall\exists}(f) = [-1, 1]$, which is a strict superset of $R_{\exists\forall}(f) = \{-1, 1\}$

(different than the linear case, where $R_{\forall\exists}(f)$ and $R_{\exists\forall}(f)$ would not agree only in the case when the latter is empty)

Dubbins example (II)

Direct computation from the ODE (no need for Taylor approximant)

- Outer-approximation of a "central trajectory" (x_c, y_c, θ_c) starting at $x = 0$, $y = 0$, $\theta = 0$, $b_1 = 0$ and $a = 0$: $x_c = t$, $y_c = 0$ and $\theta_c = 0$,
- $\frac{\partial x}{\partial t} = cos(\theta) + b_1 \in [0.989999965, 1.01]$ hence $I_{x,t} = [0, 0.494999982]$, $O_{x,t} = [0, 0.505]$,
- Similarly for the other variables: $I_{y,t} = 0$, $O_{y,t} = [-sin(0.015)/2, sin(0.015)/2] = [-1.309\ 10^{-4}, 1.309\ 10^{-4}]$ and $I_{\theta,t} = 0$, $O_{\theta,t} = [-0.005, 0.005]$,
- The Jacobian of φ with respect to x_0, y_0, θ_0, b_1 and a, satisfies a variational equation, we find:
  - $I_{x,a} = 0$, $O_{x,a} = [-6.545\ 10^{-7}, 6.545\ 10^{-7}]$, $I_{x,x_0} = O_{x,x_0} = [-0.1, 0.1]$, $I_{x,\theta_0} = 0$, $O_{x,\theta_0} = [-1.309\ 10^{-6}, 1.309\ 10^{-6}]$, $I_{x,b_1} = 0$, $O_{x,b_1} = [-0.005, 0.005]$,
  - $I_{y,a} = 0$, $O_{y,a} = [-0, 0025,\ 0.0025]$, $I_{y,y_0} = O_{y,y_0} = [-0.1, 0.1]$, $I_{y,\theta_0} = 0$, $O_{y,\theta_0} = [-0, 005, 0.005]$,
  - $I_{\theta,\theta_0} = O_{\theta,\theta_0} = [-0.01, 0.01]$, $I_{\theta,a} = 0$, $O_{\theta,a} = [0, 0.005]$,

Dubbins example (II)

Compute $R_{\exists\forall\exists}$:

$$\exists a \in [-0.01, 0.01],\ \exists x_0 \in [-0.1, 0.1],\ \exists y_0 \in [-0.1, 0.1],\\ \exists \theta_0 \in [-0.01, 0.01],\ \forall b_1 \in [-0.01, 0.01],\ \exists t \in [0, 0.5],\\ z = \varphi(t; x_0, y_0, \theta_0, a, b_1)$$

Hence, inner-approximation

Lower bound inner-approximation for x:

$$\begin{array}{ccccccc} x_c & +\underline{I}_{x,a} & +\underline{I}_{x,x_0} & +\underline{I}_{x,y_0} & +\underline{I}_{x,\theta_0} & +\overline{O}_{x,b_1} & +\underline{I}_{x,t} \\ = 0 & -0 & -0.1 & +0 & -0 & +0.005 & +0 \end{array}$$

which is equal to -0.095, and its upper bound:

$$\begin{array}{ccccccc} x_c & +\overline{I}_{x,a} & +\overline{I}_{x,x_0} & +\overline{I}_{x,y_0} & +\overline{I}_{x,\theta_0} & +\underline{O}_{x,b_1} & +\overline{I}_{x,t} \\ 0 & +0 & +0.1 & +0 & +0 & -0.005 & +0.494999982 \end{array}$$

which is equal to 0.589999982. Therefore the inner-approximation for x is equal to $[-0.095, 0.589999982]$.

Dubbins example (II)

Compute $R_{\exists\forall\exists}$:

$$\exists a \in [-0.01, 0.01],\ \exists x_0 \in [-0.1, 0.1],\ \exists y_0 \in [-0.1, 0.1],\\ \exists \theta_0 \in [-0.01, 0.01],\ \forall b_1 \in [-0.01, 0.01],\ \exists t \in [0, 0.5],\\ z = \varphi(t; x_0, y_0, \theta_0, a, b_1)$$

Hence, outer-approximation

Lower bound outer-approximation for the x:

$$\begin{array}{ccccccc} x_c & +\underline{O}_{x,a} & +\underline{O}_{x,x_0} & +\underline{O}_{x,y_0} & +\underline{O}_{x,\theta_0} & +\overline{I}_{x,b_1} & +\underline{O}_{x,t} \\ = 0 & -6.545\ 10^{-7} & -0.1 & +0 & -1.309\ 10^{-6} & +0 & +0 \end{array}$$

which is equal to -0.1000019635, and its upper bound:

$$\begin{array}{ccccccc} x_c & +\overline{O}_{x,a} & +\overline{O}_{x,x_0} & +\overline{O}_{x,y_0} & +\overline{O}_{x,\theta_0} & +\underline{I}_{x,b_1} & +\overline{O}_{x,t} \\ = 0 & +6.545\ 10^{-7} & +0.1 & 0 & +1.309\ 10^{-6} & -0 & +0.505 \end{array}$$

which is equal to 0.6050019635. Therefore the outer-approximation for x is equal to $[-0.1000019635, 0.6050019635]$.

Dubbins example (II)

Compute $R_{\exists\forall\exists}$:

$$\exists a \in [-0.01, 0.01],\ \exists x_0 \in [-0.1, 0.1],\ \exists y_0 \in [-0.1, 0.1],\ \exists \theta_0 \in [-0.01, 0.01],\ \forall b_1 \in [-0.01, 0.01],\ \exists t \in [0, 0.5],\ z = \varphi(t; x_0, y_0, \theta_0, a, b_1)$$

And...

- for y the inner-approximation $[-0.1, 0.1]$ and over-approximation $[0.1076309, 0.1076309]$,
- and for θ the inner-approximation $[-0.01,\ 0.01]$ and over-approximation $[-0.02, 0.02]$.

Very close to results obtained by quantifier elimination (Mathematica), here with a much smaller complexity.

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Problematic

Example

Inner approximate $R_{\forall\exists\forall\exists}(f) = \{z \mid \forall x_1,\ \exists x_2,\ \exists x_3,\ \forall x_4,\ \exists x_5,\ \exists x_6, z = f(x)\}$?

- Outer-approximation of each component, separately, will give an outer-approximation of $R_{\forall\exists\forall\exists}(f)$
- But not for the inner-approximation!

Idea, for "joint" inner-approximation

- Conjunction of quantified formulas for each component if no variable is existentially quantified for several components.
- Transform the quantified formula by strengthening them for that objective

For example:

$$\forall x_1,\ \forall x_2,\ \boxed{\exists x_3},\ \forall x_4,\ \forall x_5,\ \boxed{\exists x_6},\ z_1 = f_1(x_1, x_2, x_3, x_4, x_5, x_6)$$
$$\forall x_1,\ \forall x_3,\ \boxed{\exists x_2},\ \forall x_4,\ \forall x_6,\ \boxed{\exists x_5},\ z_2 = f_2(x_1, x_2, x_3, x_4, x_5, x_6)$$

General theorem

More formally... and I am not going to go through this!

Let $f : \mathbb{R}^u \to \mathbb{R}^m$ be an elementary function and $\pi^i : \{k_{2i}+1, \ldots, k_{2i+1}\} \to \{1, \ldots, m\}$ for $i = 1, \ldots, n$. Let us note, for all $i \in \{1, \ldots n\}$, $j \in \{1, \ldots, m\}$ $J^i_{E,z_j} = \{l \in \{k_{2i}+1, \ldots, k_{2i+1}\},\ \pi^i(l) = j\}$ and $J^i_{A,z_j} = \{k_{2i-1}+1, \ldots, k_{2i}\} \setminus J_{E,z_i}$.
Consider the following m quantified problems, $j \in \{1, \ldots, m\}$:

$$\forall z_j \in \boldsymbol{z}_j,\ (\forall \boldsymbol{x}_l \in [-1,1])_{l \in J^1_{A,z_j}},\ (\exists \boldsymbol{x}_l \in [-1,1])_{l \in J^1_{E,z_j}}, \ldots$$
$$(\forall \boldsymbol{x}_l \in [-1,1])_{l \in J^n_{A,z_j}},\ (\exists x_j \in [-1,1])_{l \in J^n_{E,z_j}},\ z_i = f_i(x_1, \ldots, x_{k_{2n}})$$

Then $\boldsymbol{z} = \boldsymbol{z}_1 \times \boldsymbol{z}_2 \times \ldots \times \boldsymbol{z}_n$, if non-empty, is an inner-approximation of $R_{\boldsymbol{p}}(f)$.

Example

Consider $f = (f_1, f_2) : \mathbb{R}^4 \to \mathbb{R}^2$:

$$\begin{array}{rcl} f_1(x_1, x_2, x_3, x_4) & = & 2 + 2x_1 + x_2 + 3x_3 + x_4 \\ f_2(x_1, x_2, x_3, x_4) & = & -1 - x_1 - x_2 + x_3 + 5x_4 \end{array}$$

And compute:

$$R_{\exists\forall\exists}(f) = \{z \in \mathbb{R}^2 | \exists x_1 \in [-1,1],\ \forall x_2 \in [-1,1],\ \exists x_3 \in [-1,1], \\ \exists x_4 \in [-1,1],\ z = f(x_1, x_2, x_3, x_4)\}$$

Same calculation as before, 1 component at a time: $R_{\exists\forall\exists}(f) \subseteq [-3,7] \times [-7,5]$.

For the joint inner-approximation, interpret:

$$\boxed{\exists x_1},\ \forall x_2,\ \forall x_3,\ \boxed{\exists x_4},\ z_1 = f_1(x_1, x_2, x_3, x_4)$$
$$\forall x_1,\ \forall x_2,\ \forall x_4,\ \boxed{\exists x_3},\ z_2 = f_2(x_1, x_2, x_3, x_4)$$

Empty set for z_1 already: contribution of the existentially quantified x_4 is $[-1,1]$ whereas the universally quantified x_2 and x_3 account for $[-4,4]$, which thus cannot be fully compensated

Example

Consider $f = (f_1, f_2) : \mathbb{R}^4 \to \mathbb{R}^2$:

$$\begin{array}{rcl} f_1(x_1, x_2, x_3, x_4) & = & 2 + 2x_1 + x_2 + 3x_3 + x_4 \\ f_2(x_1, x_2, x_3, x_4) & = & -1 - x_1 - x_2 + x_3 + 5x_4 \end{array}$$

And compute:

$$R_{\exists\forall\exists}(f) = \{z \in \mathbb{R}^2 | \exists x_1 \in [-1,1],\ \forall x_2 \in [-1,1],\ \exists x_3 \in [-1,1],\ \exists x_4 \in [-1,1],\ z = f(x_1, x_2, x_3, x_4)\}$$

For the joint inner-approximation, interpret:

$$\boxed{\exists x_1},\ \forall x_2,\ \forall x_4,\ \boxed{\exists x_3},\ z_1 = f_1(x_1, x_2, x_3, x_4)$$

$$\forall x_1,\ \forall x_2,\ \forall x_3,\ \boxed{\exists x_4},\ z_2 = f_2(x_1, x_2, x_3, x_4)$$

$$\begin{array}{l} z_1 = [\ z_1^c - ||\Delta_{x_1}|| + ||\Delta_{x_2,x_4}|| - ||\Delta_{x_3}||,\ z_1^c + ||\Delta_{x_1}|| - ||\Delta_{x_2,x_4}|| + ||\Delta_{x_3}||] \\ = [\ 2 \quad -2 \quad +1+1 \quad -3,\quad 2 \quad +2 \quad -1-1 \quad +3] = [-1, 5] \end{array}$$

Example

Consider $f = (f_1, f_2) : \mathbb{R}^4 \to \mathbb{R}^2$:

$$\begin{array}{rcl} f_1(x_1,x_2,x_3,x_4) & = & 2 + 2x_1 + x_2 + 3x_3 + x_4 \\ f_2(x_1,x_2,x_3,x_4) & = & -1 - x_1 - x_2 + x_3 + 5x_4 \end{array}$$

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For the joint inner-approximation, interpret:

$$\boxed{\exists x_1},\ \forall x_2,\ \forall x_4,\ \boxed{\exists x_3},\ z_1 = f_1(x_1,x_2,x_3,x_4)$$

$$\forall x_1,\ \forall x_2,\ \forall x_3,\ \boxed{\exists x_4},\ z_2 = f_2(x_1,x_2,x_3,x_4)$$

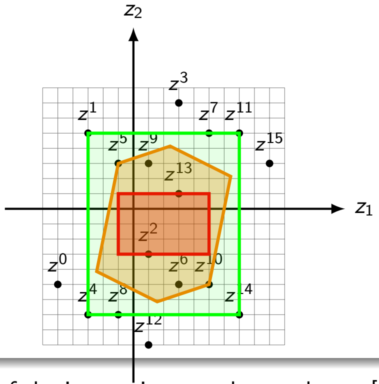
$$\begin{array}{l} z_2 = [\ z_2^c\ + ||\Delta_{x_1,x_2,x_4}|| - ||\Delta_{x_3}||,\ z_1^c\ - ||\Delta_{x_1,x_2,x_4}||\quad + ||\Delta_{x_3}||] \\ \quad = [\ \ -1\ \ +1+1+1\quad -5,\quad -1\ \ -1-1-1\ \ +5] = [-3,1] \end{array}$$

Hence $[-1,5] \times [-3,1] \subseteq R_{\exists\forall\exists}(f) \subseteq [-3,7] \times [-7,5]$.

Example, in picture

$$\begin{aligned} f_1(x_1, x_2, x_3, x_4) &= 2 + 2x_1 + x_2 + 3x_3 + x_4 \\ f_2(x_1, x_2, x_3, x_4) &= -1 - x_1 - x_2 + x_3 + 5x_4 \end{aligned}$$

$$R_{\exists\forall\exists}(f) = \{z \in \mathbb{R}^2 | \exists x_1,\ \forall x_2,\ \exists x_3, \exists x_4,\ z = f(x_1, x_2, x_3, x_4)\}$$



(some particular points of the image; inner and outer boxes $[-1, 5] \times [-3, 1]$ and $[-3, 7] \times [-7, 5]$; polyhedron lying in between is the exact robust image)

Last application: Dubbins!

Space relaxation

$$R_{\exists\forall\exists}(\varphi) = \{(x, y, \theta) \mid \exists a \in [-0.01, 0.01],\ \exists x_0 \in [-0.1, 0.1],\\ \exists y_0 \in [-0.1, 0.1],\ \exists \theta_0 \in [-0.01, 0.01],\ \forall b_1 \in [-0.01, 0.01],\\ \exists t \in [0, 0.5],\ \exists \delta_2 \in [-1.309\ 10^{-4}, 1.309\ 10^{-4}],\ \exists \delta_3 \in [-0.005, 0.005],\\ (x, y, \theta) = \varphi(t; x_0, y_0, \theta_0, a, b_1) + (0, \delta_2, \delta_3)\}$$

Outer-approximation

$$R_{\exists\forall\exists}(\varphi) \subseteq [-0.1000019635, 0.6050019635] \times\\ [0.1077618, 0.1077618] \times [-0.025, 0.025]$$

Last application: Dubbins!

$$R_{\exists\forall\exists}(\varphi) = \{(x, y, \theta) \mid \exists a \in [-0.01, 0.01],\ \exists x_0 \in [-0.1, 0.1],\ \exists y_0 \in [-0.1, 0.1],\ \exists \theta_0 \in [-0.01, 0.01],\ \forall b_1 \in [-0.01, 0.01],\ \exists t \in [0, 0.5],\ \exists \delta_2 \in [-1.309\ 10^{-4}, 1.309\ 10^{-4}],\ \exists \delta_3 \in [-0.005, 0.005],\ (x, y, \theta) = \varphi(t; x_0, y_0, \theta_0, a, b_1) + (0, \delta_2, \delta_3)\}$$

For the inner-approximation, interpret:

$$\forall a,\ \forall y_0, \forall \theta_0,\ \boxed{\exists x_0},\ \forall b_1,\ \forall \delta_2,\ \forall \delta_3,\ \boxed{\exists t},\ x = \varphi_x(t; x_0, y_0, \theta_0, a, b_1)$$

$$\forall a,\ \forall x_0, \forall \theta_0,\ \boxed{\exists y_0},\ \forall b_1,\ \forall \delta_3,\ \forall t,\ \boxed{\exists \delta_2},\ y = \varphi_y(t; x_0, y_0, \theta_0, a, b_1) + \delta_2$$

$$\forall x_0,\ \forall y_0, \boxed{\exists \theta_0,\ \exists a},\ \forall b_1,\ \forall \delta_2,\ \forall t,\ \boxed{\exists \delta_3},\ \theta = \varphi_\theta(t; x_0, y_0, \theta_0, a, b_1) + \delta_3$$

$$[-0.0949993455, 0.5899993275] \times [-0.0925, 0.0925] \times [-0.01, 0.01] \subseteq R_{\exists\forall\exists}(\varphi)$$

(timeout using quantifier elimination under Mathematica)

Thanks!

Any questions?